International Journal of Theoretical Physics, Vol. 31, No. 12, 1992

Experimental Facts and Gravitational Energy in General Relativity

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Received November 28, 1991

It is shown that the nonrotating coordinates wherein the energy-momentum is globally conserved share the experimental features of the inertial frames. The falling of matter in a spherically symmetric gravitational field is studied in the light of the energy-momentum conservation valid in these coordinates.

1. INTRODUCTION

The difficulties of a theory of gravitation may be traced to the reduced body of experimental data at our disposition. The motion of planets and light rays in the spherically symmetric gravitational fields of astronomical bodies and the falling of bodies toward the earth are the only physical phenomena that open a very narrow window to gravitation. The weakness of the gravitational field prevents experimentation with artificially created fields. Would it have been possible to elaborate Maxwell's theory by experimenting only in the Coulomb field of spherical conductors?

Extremely limited in our experimental possibilities, we have to act as the paleontologist who rebuilds the whole body of an antediluvian animal from a bone fragment. In this situation we cannot disregard even one of the experimental facts that nature grants us. Among them, there are the experimental facts that reveal the existence of a preferred class of coordinates: the inertial frames of Newton's mechanics (NM) and special relativity (SR), wherein stars, pendulums, and water buckets show special behavior.

According to Mach's and Einstein's philosophy, the existence of a preferred class of frames results from the energy-momentum distribution

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throughout spacetime. Intrinsically all the frames are equivalent, but not all of them are equally related to the matter actually existing in the universe. NM and SR admit a preferred realization of their respective invariance group, but lacking any bearing on the energy-momentum distribution, they are unable to provide a theoretical way for its differentiation. The existence of a preferred class of frames is therefore in the pre-general relativistic theories a well-founded experimental fact that lacks a theoretical explanation. On the other hand, general relativity (GR) is essentially related to the energy-momentum distribution, but due to the absence of global "inertial" frames, the Mach principle remains in GR a philosophical idea deprived of physical content.

The energy-momentum tensor is assumed in GR to be only locally conserved in the geodesic systems of coordinates. In an attempt to restore the global characteristic of energy-momentum conservation, Einstein and his continuators looked for a kind of energy excluded from the nonconserved tensor. This external kind of energy-momentum is intended to complement the energy-momentum tensor to a quantity with globally vanishing ordinary divergence.

This global energy-momentum conservation is required to hold in all systems of coordinates. For mathematical reasons a symmetric tensor is precluded from satisfying a global ordinary divergencelessness condition valid in all coordinate systems. Hence, the reluctance to accept a physical law valid in a preferred system of coordinates leads paradoxically to acquiescence with a nontensorial gravitational energy-momentum whose dependence on the coordinates annuls any apparent gain in covariance.

In previous papers (Nissani and Leibowitz, 1988, 1989*a*,*b*, 1990, 1991; Carmeli *et al.*, 1990) we have shown the existence of preferred frames, the nonrotating systems of coordinates, wherein the energy-momentum is globally conserved, and the stars, neglecting gravitational effects, move with constant velocity (two of the principal characteristics of the inertial frames). These coordinate systems have been the object of some (unpublished) criticism. In the interior of a rotating galaxy, it was said, the nonrotating coordinates have to rotate together with any one of the stars to provide for the energy-momentum conservation. Certainly, this is so if the gravitational energy-momentum is excluded from the conserved energy-momentum tensor. But one of the consequences of the existence of the nonrotating frames is precisely that gravitational energy may, and must, be incorporated, together with all the other members of the family, in the energy-momentum tensor. For this purpose a tensorial expression for the gravitational energy-momentum is needed. Such tensorial expression was proposed in Nissani and Leibowitz (1991).

A suitable gravitational energy-momentum tensor has to allow for the existence of physically acceptable preferred frames. Namely, the nonrotating frames have to be specializable to share the experimental characteristics of the inertial frames. To show that the proposed tensorial expression of the gravitational energy-momentum tensor satisfies this claim is the main purpose of the present work.

In the next section we describe and classify the experimental characteristics of the inertial frames. In Sections 3 and 4 the nonrotating coordinates are examined in the light of the above experimental characteristics of the inertial frames. In Section 5 the repercussion of the definition of gravitational energy-momentum on the adaptability of the nonrotating coordinates to the static character of the spacetime is discussed.

In Section 6 some of the experimental requirements to be satisfied for the gravitational energy are specified. In Section 7 the gravitational energy-momentum tensor is briefly described. In Section 8 we discuss the splitting in curved spacetime of the spatial integral of the energy into three values of physical significance. In Section 9 these three values of the integral of the gravitational energy in the surroundings of a star are calculated in static nonrotating coordinates, and are shown to satisfy the integral requirements pointed out in Section 6.

In Section 10 we discuss two different ways of measuring energy-momentum. The ensuing physical quantities will be respectively called the local and the universal energy-momentum. With the local energy-momentum the rest mass of a particle is an invariant of the motion. With the universal definition, on the contrary, it undergoes a gravitational red shift. In Section 11 the falling of test matter in a spherically symmetric gravitational field is studied by balancing gravitational against kinetic energy. It is shown that using the universal definition for the energy-momentum tensor yields results in total accordance with experience. Finally, Section 12 is devoted to remarks and conclusions.

2. THE EXPERIMENTAL FACTS

NM and SR accept the existence of a preferred realization of their respective invariance group—the inertial frames. Its existence is supported by an abundance of terrestrial experiments and astronomical observations. As experimental facts they have to be taken into account by any physical theory concerned with coordinate systems. We will classify them, for the convenience of the ensuing discussion, into the following six sets.

EF1. The local experiments that do not involve gravitation. They are satisfactorily explained by SR, and also by GR by means of the locally geodesic coordinates. In spite of their local character, they point to the same frames where distant celestial bodies show special behavior.

EF2. The local experiments that do involve gravitation, e.g., Newton's water bucket or Foucault's pendulum. They cannot be explained by means of the geodesic coordinates, since in the absence of gravitation a pendulum is not expected to oscillate and the water will hardly enter the bucket. They also point to the frames wherein stars and galaxies are in constant-velocity motion. Their explanation demands the existence of nongeodesic preferred frames.

EF3. The nonlocal observed facts such as the behavior of stars and galaxies. In an inertial frame, and when gravitational effects can be neglected, they are at rest or in motion at constant velocity. On the other hand, in a "rotating" frame they perform an ordered simultaneous rotation. Observed from the earth, they describe elliptic paths synchronized with the rotation of the earth around the sun. Confined to only local preferred coordinates, we have no means to account for these facts that embrace the whole observable universe.

EF4. Experiments that show the physical character of the preferred frames. The constancy of spectral shifts and measured angles indicates that the uniform velocity exhibited by the stars in these coordinates is a physical fact rather than a coordinate effect. The preferred frames appear as physically measurable coordinates that cannot be considered as mere labels.

The possibility of establishing the preferred coordinates by measurement requires a time-independent metric. The possibility of the existence of such a metric relies on the characteristics of the energy-momentum distribution. The slow velocities of stars and galaxies with respect to their neighbors characterize our universe as a quasistatic one. This allows the existence of coordinates with a nearly time-independent metric and orthogonal time coordinate. All this is specially true when dealing with the idealized configuration of an isolated body. Hence, according to the experimental results, the preferred coordinates are adaptable to the static characteristics of spacetime.

EF5. The mass of local experiments and astronomical observations that support the energy-momentum conservation as holding true in the same frames wherein the special behavior of pendulums, water buckets and stars takes place. The uniform velocity of stars and galaxies is by itself a manifestation of this conservation law.

When gravitational effects are not negligible, the energy-momentum conservation demands a suitable definition of gravitational energy. All the attempts to extend the energy-momentum conservation to a law valid in all coordinate systems have failed with regard to the physical nature of the ensuing gravitational energy.

EF6. The Lorentz invariance of the Maxwell equations that implies the Lorentz group as a link between the preferred frames.

In the next section we define a class of general relativistic preferred frames. We do this by means of a variational demand on the components of the energy-momentum tensor. These preferred frames will be shown to be compatible with the above six sets of experimental facts.

3. VARIATIONAL VERSION OF THE MACH PRINCIPLE

In pursuit of a class of coordinate systems linked to the energy-momentum distribution, one is led to resort to a noncovariant requirement associated with the energy-momentum tensor. Accordingly, we will look for the coordinate systems wherein the integrals of the components of the energy-momentum tensor density over an arbitrary four-dimensional volume of spacetime attain a stationary value.

To this purpose we define the following ten scalar actions as functionals of the energy-momentum tensor components and four scalar functions $\phi^{(\mu)}$ of the coordinates:

$$I^{(\mu)(\nu)} = \int_V (-g)^{1/2} T^{\alpha\beta} \phi^{(\mu)}_{,\alpha} \phi^{(\nu)}_{,\beta} \, d^4x \tag{1}$$

By varying the functions $\phi^{(\mu)}$ and requiring stationariness, one finds the following four Lagrange equations:

$$((-g)^{1/2} T^{\alpha\beta})_{,\alpha} \phi^{(\nu)}_{,\beta} + (-g)^{1/2} T^{\alpha\beta} \phi^{(\nu)}_{,\alpha\beta} = 0 \tag{2}$$

for the four functions $\phi^{(\mu)}$. Hence, equation (2) defines the class of coordinates wherein the integrals of the components of the energy-momentum tensor through an arbitrary four-dimensional volume attain a stationary value. Notice that for an antisymmetric tensor, $T^{\alpha\beta} = -T^{\beta\alpha}$, equation (2) is satisfied for any arbitrary function $\phi^{(\mu)}$ provided that

$$T^{\alpha\beta}_{;\alpha} = 0$$

and for none otherwise. If the metric substitutes for the energy-momentum tensor, equation (2) will define the harmonic coordinates (Fock, 1964).

In the particular case of isolated massive points, such as stars or galaxies at great distances from each other, and neglecting any gravitational contribution, the energy-momentum tensor may be written

$$T^{\alpha\beta} = \sum_i \frac{1}{(-g)^{1/2}} \prod_j \delta[x^j - x^j_i(x^0)] M_i U^\alpha_i(x^0) U^\beta_i(x^0) \tag{3}$$

where i runs over stars, j over the three spatial coordinates, x^0 is the time coordinate, M_i are the time components of a covariant vector that in the Schwarzschild coordinates of the respective star take the values of the Schwarzschild stellar masses, and U_i are the stellar four-vector velocities.

Using (3) in (2) and putting

$$V^{\alpha}=U^{\beta}\phi^{(\alpha)}_{,\beta} \tag{4}$$

where V stands for the velocities of the stars in the selected coordinate systems that satisfy equation (2), and accepting the conservation of the stars' masses expressed by

$$\left\{\sum_{i}\frac{1}{(-g)^{1/2}}\prod_{j}\delta[x^{j}-x^{j}_{i}(x^{0})]M_{i}U^{\beta}_{i}(x^{0})\right\}_{,\beta}=0$$

one finds

$$dV^{\alpha}_{i}/dx^{0}=0 \tag{5}$$

Therefore, equation (2) defines, in strict general relativistic terms, the coordinate systems where the distant stars are at rest or in constant-velocity motion. These coordinates have been named the nonrotating coordinates since they share with Newton's fixed-stars frames, selected by Newton's water bucket, the most conspicuous characteristics.

Furthermore, denoting by g' and T' the values of the metric and energy-momentum tensors in the nonrotating coordinates, equation (2) becomes

$$((-g')^{1/2}T'^{\alpha\beta})_{,\alpha}=0 \tag{6}$$

Namely, in the nonrotating coordinates the energy-momentum tensor density satisfies a global continuity equation. They satisfy, therefore, the experimental facts EF3 and EF5 of the preceding section that distinguish the inertial frames. Notice that if the energy-momentum tensor is replaced by the metric, equation (6) becomes the deDonder condition of the harmonic coordinates.

In the next section it will be shown that the nonrotating coordinates can be specialized to be locally geodesic with respect to any given observer. It is precisely the existence of both geodesic and nongeodesic nonrotating coordinates that makes it possible, at least in principle, to explain the sets EF1 and EF2 of local experimental facts.

4. THE LOCAL EXPERIMENTAL FACTS

So far, we have not made use of the covariant divergencelessness of the energy-momentum tensor. We now have to resort to it to include in the same preferred coordinate systems, together with the global energy-momentum conservation and the special behavior of the distant stars (EF3 and EF5), the local special relativistic form of the laws of physics (EF1). For it to be possible to do this, the nonrotating coordinates have to include locally geodesic coordinates with respect to any given observer.

Note that in locally geodesic coordinates, equation (6) is equivalent to a covariant divergencelessness condition:

$$T'^{\alpha\beta}_{;\alpha}=0 \tag{7}$$

Therefore, the covariant conservation of the energy-momentum tensor is a necessary condition for the nonrotating coordinates to include geodesic coordinates with respect to any given observer. That this condition is also sufficient was illustrated in detail in Nissani and Leibowitz (1989). Namely, for any given observer there is a subclass of the geodesic coordinates wherein the energy-momentum tensor is globally conserved and the stars move with constant velocity. It is in these frames that the local experimental facts EF1, EF3, and EF5 find their explanation. They will be referred to as the geodesic nonrotating coordinates.

It is easy to see from equation (2) that the internal group of the nonrotating coordinates is defined by

$$T^{\alpha\beta}\phi^{(\nu)}_{,\alpha\beta}=0 \tag{8}$$

It is a broad subgroup of the general mapping group that includes the Lorentz group. The nonrotating coordinates defined by equation (2) constitute its preferred representation.

The internal group of the geodesic nonrotating frames of a given observer is the subgroup of the group (8) made up by the transformations that are locally Lorentzian. This is in agreement with the fundamental role of the Lorentz transformation in the local experimental facts (EF6). On the other hand, the nonlocally Lorentzian transformations of the group link geodesic to nongeodesic nonrotating coordinates. The existence of the nongeodesic nonrotating coordinates, i.e., preferred frames where gravitational effects are present, makes possible the explanation of the experimental facts that involve gravitation (EF2).

5. STATIC NONROTATING COORDINATES

As was indicated in Section 2, there is strong experimental evidence of the (nearly) static characteristic of the inertial frames (EF4). It is therefore natural to demand the existence of nonrotating coordinates that share this characteristic when one deals with static configurations. Obviously, in a totally static universe one has to expect static solution of equation (2). But, clearly, we are not interested in the physics of a dead universe with an energy-momentum tensor totally independent of time. The question arises of the extent to which the nonrotating coordinates may continue to be assumed static for the study of the dynamics of test bodies in a static gravitational field. The answer seems to be that it rests on the appropriate definition of the gravitational energy-momentum.

To clarify the effect of the definition of gravitational energy-momentum on the physical properties of the nonrotating frames, consider the extreme case of a gravitational energy-momentum which is by definition everywhere null; or, what is for this case the same, a gravitational energy-momentum external to the energy-momentum tensor. Now, assume an isolated test body in an arbitrary gravitational field. The energy-momentum tensor in the vicinity of the test body would be, in this case, of the form described by equation (3). Hence, if the conservation of the scalar rest mass of the body is assumed, equation (5) holds true. Therefore, the test body would be at rest or with constant velocity with respect to the nonrotating coordinates. Namely, the nonrotating coordinates would escort any body in its motion. They by no means would exhibit the physical characteristics of the inertial frames.

Only a suitable expression for the gravitational energy-momentum as an integral part of the conserved energy-momentum tensor can lead to physically acceptable nonrotating coordinates. In the following, a tensorial expression for the gravitational energy-momentum compatible with the existence of static nonrotating coordinates will be considered.

6. GLOBAL ENERGY-MOMENTUM CONSERVATION AND GRAVITATIONAL ENERGY

As pointed out in the Introduction, in the conventional general relativistic approach the energy-momentum tensor is conserved only locally in the geodesic systems of coordinates. It is only in these locally preferred systems of coordinates that the covariant divergencelessness of the energy-momentum tensor reads as an ordinary divergencelessness, i.e., as a continuity equation. This lack of global conservation led to the search for a kind of energy external to the energy-momentum tensor which is assumed to be of gravitational nature. This external kind of energy is intended to complement the energy-momentum tensor, yielding a globally conserved quantity. Hence, the energy-momentum tensor is understood to be built up of only "matter" energy.

The resulting globally conserved quantity, the sum of matter and gravitational energy-momentum, is, in the conventional approach, required to satisfy a continuity equation in *all* systems of coordinates. For mathematical reasons a symmetric tensor cannot satisfy this requirement. It is necessary, therefore, to give up the symmetry or the tensorial nature of the conserved quantity. Either of these renunciations demands a very high price, in philosophical as well as in practical currency, in terms of our understanding of physics. The symmetry of the conserved energy-momentum quantity is necessary for the conservation of the angular momentum, whereas relinquishing the tensorial character (the generally favored way) results in an anomalous nonlocalized kind of energy.

The existence of the nonrotating frames whereby the energy-momentum tensor satisfies a global continuity equation precludes the interchange between a kind of energy included in the tensor and another kind of energy excluded from it. [For the interested reader this was illustrated by means of a thought experiment in Nissani and Leibowitz (1988).] Hence, there is no place, and also no more need, for a kind of energy excluded from the energy-momentum tensor. The gravitational energy should therefore be tensorial and included, together with all the other manifestations of the energy, in the energy-momentum tensor. Accordingly, the Einstein field equation

$$G^{\alpha\beta} = \kappa T_T^{\alpha\beta} \tag{9}$$

now has to be interpreted with

$$T_T^{\alpha\beta} = T_M^{\alpha\beta} + T_G^{\alpha\beta} \tag{10}$$

where T_M and T_G are the matter and the gravitational energy-momentum tensors, respectively.

The goal is now to find the appropriate expression for the gravitational energy-momentum tensor T_G. Clearly, the final test to establish the appropriateness of the expression is whether it fits the experimental facts. In practice, the experimental facts involving gravitational energy are limited to the movement of bodies in astronomical gravitational fields. In a conservative system, such as the inertial frames in NM or the nonrotating coordinates in GR, the change in "matter" energy-momentum of a falling body should be equal and of opposite sign to the change in the gravitational energy-momentum of the field.

Calculated by balancing gravitational and kinetic energy, the Newtonian gravitational energy in the surroundings of a star is given by

$$E_{\rm N} = -\tfrac{1}{2} M^2/R \tag{11}$$

where M and R are the star mass and radius, respectively. Consequently, for an ordinary star we should expect a relativistic value highly approximated to the Newtonian value when calculated in a nonrotating system of coordinates with respect to which the star is at rest. Therefore, we have to demand from the suitable expression of the gravitational energy T_G^{00} that

$$E_G^0 = 4\pi \int_{R \gg 2M}^{\infty} T_G^{00}\, dr \approx -\tfrac{1}{2} M^2/R \tag{12}$$

while for a black hole, with escape velocity equal to the velocity of light, we should have

$$E_G^0 = 4\pi \int_{R=2M}^{\infty} T_G^{00}\, dr = -\infty \tag{13}$$

and for the velocity of a body falling from infinity we should attain, when calculated by balancing gravitational and kinetic energy, a highly approximated value to the geodesic path velocity v_E,

$$v_E = 2^{1/2} MG/r \tag{14}$$

which is also the expression of the corresponding Newtonian velocity if r is identified with the Newtonian distance.

In this approach the gravitational energy-momentum is part of the conserved tensor that determines the nonrotating coordinates. Hence, the definition of T_G affects the determination of the preferred systems of coordinates. We should therefore demand that the definition of gravitational energy-momentum lead to physically acceptable nonrotating coordinates. According to the set EF4 of experimental facts and what was said in the preceding section, in a static spacetime the nonrotating frames should include static coordinates, i.e., frames with a time-independent metric and an orthogonal time coordinate,

$$\partial g^{\alpha\beta}/\partial x^0 = 0, \qquad g^{0i} = 0 \tag{15}$$

In the next section we will show the existence of a tensor that fulfills equations (12) and (13), with the integrals carried out in a coordinate system wherein the metric satisfies equation (15). In Section 11 the free falling of matter is satisfactorily studied by means of this tensor in the light of the energy-momentum conservation. It appears therefore as a suitable tensorial expression for the gravitational energy-momentum.

7. THE GRAVITATIONAL ENERGY-MOMENTUM TENSOR

With the aid of an orthonormal tetrad ϕ_a^α

$$\phi_a^\alpha \eta^{ab} \phi_b^\beta = g^{\alpha\beta} \tag{16}$$

where η^{ab} is the Minkowskian matrix, the Ricci tensor may be written

$$R_{\beta\gamma} = (P^\alpha_{\beta\alpha;\gamma} - P^\alpha_{\beta\gamma;\alpha}) + (P^\alpha_{\delta\gamma} P^\delta_{\beta\alpha} - P^\alpha_{\delta\alpha} P^\delta_{\beta\gamma}) \tag{17}$$

with the tensor P defined by

$$P^{\alpha}_{\beta\gamma} = \phi^{\alpha}_{a}\phi^{a}_{\beta;\gamma} \tag{18}$$

Equation (17) shows the decomposition of the Ricci tensor into two tensors that are the covariant rotor and the commutator of the tensor P.

This decomposition of the Ricci tensor induces, in turn, a decomposition of the Einstein tensor,

$$G_{G\alpha\beta} = R_{G(\alpha\beta)} - \tfrac{1}{2} g_{\alpha\beta} R^{\rho}_{G\rho} \tag{19}$$

$$G_{M\alpha\beta} = R_{M(\alpha\beta)} - \tfrac{1}{2} g_{\alpha\beta} R^{\rho}_{M\rho} \tag{20}$$

with

$$R_{G\beta\gamma} = (P^{\alpha}_{\beta\alpha;\gamma} - P^{\alpha}_{\beta\gamma;\alpha}) \tag{21}$$

and

$$R_{M\beta\gamma} = (P^{\alpha}_{\delta\gamma} P^{\delta}_{\beta\alpha} - P^{\alpha}_{\delta\alpha} P^{\delta}_{\beta\gamma}) \tag{22}$$

Now, we assume

$$T^{\alpha\beta}_{G} = (1/\kappa)_{G} G^{\alpha\beta} \tag{23}$$

as the definition of the gravitational energy-momentum tensor. This assumption, which may seem here somewhat arbitrary, will be justified later by showing that it satisfies the experimental demands described in the previous section [equations (12) to (15)].

Now, from equations (9), (10), and (23) one obtains

$$G^{\alpha\beta}_{M} = \kappa T^{\alpha\beta}_{M} \tag{24}$$

Equations (23) and (24) are therefore the tetrad form of the Einstein field equation. Regarding the gravitational and matter energy-momentum as data, and taking into account the Bianchi identities, they constitute a set of 16 independent equations for the 16 components of the tetrad.

It has to be pointed out that for a given metric there are infinite different possible partitions of the total energy-momentum into its matter and gravitational parts. This partition is invariant under a global Lorentz transformation of the tetrad, but it varies under a local Lorentz transformation. This is because the gravitational energy-momentum is not a function of the metric, which is Lorentz invariant, but of the orthonormal tetrad field. Therefore, this tetrad, and not the metric, has to be regarded as the fundamental element of the gravitational field. It will be called the fundamental tetrad.

Nevertheless, we assume the existence of a specific amount of gravitational energy-momentum associated with the curvature of spacetime. It will be called the "gravitational energy-momentum of curvature" to distinguish it from other possible manifestations of gravitational energy-momentum, e.g., gravitational waves. To this category, gravitational energy of curvature, belongs the gravitational energy in the neighborhood of a star calculated by equation (12).

This implies that we have to find a way to single out the tetrad field associated with the energy of curvature. We do this in Section 9 for the gravitational field of a spherically symmetric source. But first we clarify some concepts that we will need later.

8. THE INTEGRAL ENERGY

There are three different integrals of the energy contained in a given spatial volume V that we will need to consider in the following sections. In all the cases they are understood to be carried out in nonrotating coordinates.

The first integral, the conserved energy E^0, is the time component of an affine contravariant vector,

$$E^0 = \int_V (-g)^{1/2} T^{00}\, d^3x \tag{25}$$

Its physical significance arises from being, in the nonrotating coordinates and when T is the total energy-momentum tensor, a conserved quantity. It will be called the "conserved energy" even when T is a partial energy-momentum tensor, since it expresses a contribution to a conserved quantity.

The second integral is the scalar integral energy E,

$$E = \int_V (-g)^{1/2} \Phi_\alpha T^{\alpha 0}\, d^3x \tag{26}$$

where Φ is a unitary time vector. For a static environment, taking the vector Φ normal to the hypersurface defined by the spatial static coordinates, E is a well-defined scalar parameter of the system.

Finally, the third integral is the time component of an affine covariant vector,

$$E_0 = \int_V (-g)^{1/2} T_0^0\, d^3x \tag{27}$$

Its importance arises from being, when the integral is carried out in static coordinates, the mass parameter M of the Schwarzschild solution. Notice that the three integrals (25)–(27) are invariant, except for a trivial change of scale, under transformations among static coordinates.

9. GRAVITATIONAL ENERGY IN THE SURROUNDINGS OF A STAR

Now, that we are equipped with the necessary tools, let us attempt to evaluate the amount of gravitational energy of curvature in the particular case of a spherically symmetric source of radius R and Schwarzschild mass M,

$$M = 4\pi \int_0^R T^0_{T0}\, dr \tag{28}$$

with vanishing total energy-momentum outside the source, i.e., $T_T = T_M + T_G = 0$ for $r > R$. This last requirement is simply a mathematical idealization of the physical situation intended to take advantage of the Schwarzschild solution

$$g^{\alpha\beta} = \text{diag}[1/\Lambda, -\Lambda, -r^{-2}, -r^{-2}\sin^{-2}\theta], \qquad \Lambda = 1 - 2MG/r \tag{29}$$

Notice that the integral (28), which is of the same kind as integral (27), is the usual expression of the Schwarzschild mass (Landau and Lifshitz, 1975), but with the total energy-momentum tensor, including gravitational contribution, as integrand.

It may be convenient to consider the possible presence of distant stars at rest. Their presence will illustrate the concept of nonrotating coordinates. They are assumed to be sufficiently distant to allow the use of the Schwarzschild solution in the neighborhood of the source.

Up to now we have only the 10 equations (16) that relate the 16 components of the tetrad with the metric (29). They have an infinite set of solutions, which account for all the possible values $T_M = -T_G$, namely, $T_T = 0$. However, we have assumed the existence of a specific form of gravitational energy-momentum associated with the curvature of spacetime, which is expected to be uniquely determined by the metric. It was shown in Nissani and Leibowitz (1991) that in this particular case the following tetrad is singled out by symmetry considerations:

$$\begin{aligned}
\phi^\alpha_0 &= (\Lambda^{-1/2}, 0, 0, 0)\\
\phi^\alpha_1 &= (0, \Lambda^{1/2}\sin\theta\cos\phi, r^{-1}\cos\theta\cos\phi, -r^{-1}\sin^{-1}\theta\sin\phi)\\
\phi^\alpha_2 &= (0, \Lambda^{1/2}\sin\theta\sin\phi, r^{-1}\cos\theta\sin\phi, -r^{-1}\sin^{-1}\theta\cos\phi)\\
\phi^\alpha_3 &= (0, \Lambda^{1/2}\cos\theta, -r^{-1}\sin\theta, 0)
\end{aligned} \tag{30}$$

This can be justified by the following reasons:

(a) In flat spacetime, with $M=0$, equations (30) give, up to a physically inconsequential global Lorentz transformation, the general expression of a parallel orthonormal tetrad field, i.e., with vanishing covariant derivatives, $\phi_b{}^\alpha{}_{;\beta}=0$. However, with $M>0$ the tetrad deviates with parallelism due to the curvature of spacetime.

(b) Using equation (30) in the definition (23) of the gravitational energy-momentum tensor, its ensuing nonvanishing components are

$$\begin{aligned} T^0_{G0} &= -\frac{1}{\kappa}(1-\Lambda^{1/2})^2 r^{-2} \\ T^1_{G1} &= -\frac{1}{\kappa}[(1-\Lambda^{1/2})^2 r^{-2}+(1-\Lambda^{-1/2})\cdot 2MG/r^3] \\ T^2_{G2} &= T^3_{G3} = -\frac{1}{\kappa}(1-\Lambda^{-1/2})MG/r^3 \end{aligned} \tag{31}$$

All of them, as should be expected, vanish in flat spacetime ($M=0$). The resulting nonvanishing value of the gravitational energy-momentum in curved spacetime ($M>0$) may therefore be traced to the deviation of the tetrad from parallelism due to curvature.

(c) The resulting gravitational energy-momentum, which is diagonal and with $T^{22}_G=T^{33}_G$ in the Schwarzschild coordinates, fits in the symmetry of the curvature. It shows two preferred directions, temporal and radial, and one preferred plane, normal to the radius. In addition it vanishes for $r=\infty$, as expected in an asymptotically flat spacetime.

As was indicated in the last section, the integrals (25)–(27) are invariant under transformations between static coordinates. Hence, assuming the existence of static nonrotating coordinates, they may be performed in Schwarzschild coordinates. Taking the expression (31) of the gravitational energy-momentum of curvature and performing the integrals from the surface of the source to infinity, one obtains

$$E^0_G=2M\left(\ln\frac{2\Omega}{1+\Omega}+\frac{1-\Omega}{2(1+\Omega)}\right), \qquad \Omega=(1-2MG/R)^{1/2} \tag{32}$$

for the conserved gravitational energy, whereas the amount of scalar gravitational energy is given by

$$E_G=-M\frac{1-\Omega}{1+\Omega} \tag{33}$$

and the contribution of the external gravitational energy to the Schwarzschild mass of the system is

$$M_G = 2M\left(\ln\frac{1+\Omega}{2} + \frac{1-\Omega}{2(1+\Omega)}\right) \tag{34}$$

It is easy to verify that for $2MG/R \ll 1$ one has

$$E_G^0 \approx E_G \approx M_G \approx E_N = -\tfrac{1}{2}GM^2/R \tag{35}$$

In the case of a regular star such as the sun, with $R/2MG = 2.32 \times 10^5$, one obtains from equations (32)–(34) the following relativistic values of the gravitational energy of curvature outside the star per unit of mass:

$$E_G^0/M \approx -1.077589 \cdots \times 10^{-6}$$

$$E_G/M \approx -1.077588 \cdots \times 10^{-6}$$

$$M_G/M \approx -1.077587 \cdots \times 10^{-6}$$

which agree with the Newtonian value

$$E_N/M \approx -1.077586 \cdots \times 10^{-6}$$

up to the sixth digit. For a black hole one has

$$E_G^0/M = -\infty$$

Hence, the proposed tensorial expression for the gravitational energy-momentum satisfies the requirements (12) and (13) when one assumes,

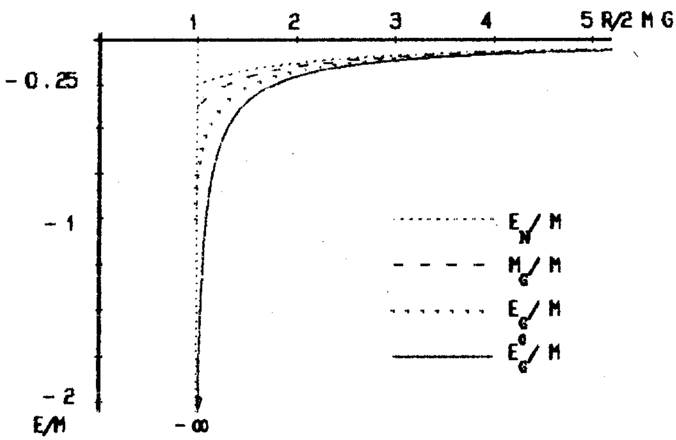

Fig. 1. Relativistic and Newtonian gravitational energies for unit stellar mass versus the ratio of stellar radius to Schwarzschild radius.

according to the demand (15), that the nonrotating coordinates can be specialized to be adapted to the static characteristic of spacetime.

The values for a unit of stellar mass of the Newtonian and relativistic gravitational energies outside the star, E_N/M, M_G/M, E_G/M, and E_G^0/M, are plotted in Figure 1 against the ratio of the radius of the star to its Schwarzschild radius, $R/2MG$. The plot includes only the first few Schwarzschild radii showing the rapid convergence of the relativistic values to the Newtonian value.

10. LOCAL VERSUS UNIVERSAL MEASUREMENT OF FREQUENCY AND ENERGY

Frequency is usually defined as a quantity measured by comparison with a local standard clock. This clock may be represented by the frequency of a photon emitted by an atom at rest near the observer. This kind of frequency measurement by comparison with a nearby clock will be called in the following a "local measurement" and the resulting quantity a "local frequency." However, one can also carry out a frequency measurement by comparison with the frequency of a universal standard photon, such as the background radiation or a photon coming from a specific star or galaxy. This second method of frequency measurement will be called a "universal measurement" and the corresponding physical quantity a "universal frequency."

The local and the universal frequencies are different physical quantities. The local frequency of a specific orbital transition is the same no matter where the emitting atom is placed, whereas the universal frequency reveals the gravitational redshift. (We omit here dealing with the Doppler shift because it can be eliminated by weighting measurements with photons coming from different directions.)

The trouble with the local frequency arises when one compares two local frequencies at different points in a gravitational field. One finds that of two clocks ticking at the same local rhythm, one is going faster than the other. On the other hand, if regulated to the same universal frequency, the two clocks compare satisfactorily.

Similarly, one can define a "local" and a "universal" energy-momentum. In the first case the rest mass of a particle is measured by a comparison with the energy of a locally emitted photon, while in the second, it is measured by comparison with the energy of a universal standard photon. Clearly, the local rest mass is an invariant of the movement, whereas the universal rest mass undergoes a gravitational redshift when the particle moves toward a massive body.

In the deduction of equation (5) use was made of the invariance of the stellar rest masses. This implies the use of the local definition of energy-momentum. But it still remains approximately correct with the universal definition of the energy-momentum if gravitational effects can be neglected, as is the case with isolated stars. All the other results of the preceding sections are independent of which of the definitions of energy is chosen.

In the next section we will compare the energy of falling matter at different points along its path. It is equivalent to the comparison of frequencies at different points in a gravitational field. It is not surprising, therefore, that the local energy will be inadequate, whereas the universal energy will be satisfactory. As will be shown, when calculating the velocity of falling of test matter, by balancing gravitational and kinetic energy with the universal definition of energy, the correct result is attained, namely, a result closely approximated to the Newtonian value. It is left for the interested reader to verify that the same calculation with the local energy leads to absurd consequences. This seems to imply that the energy-momentum tensor appearing in the Einstein field equation has to be of universal nature.

11. THE FALL OF TEST MATTER

In the conventional general relativistic approach, a test body in a gravitational field moves along a geodesic line. This geodesic path results from the covariant divergencelessness of the matter energy-momentum tensor together with the invariance of the scalar rest mass (Einstein and Grommer, 1927; Infeld and Schild, 1949).

In our approach, however, the covariant-divergenceless quantity is not the matter, but the total energy-momentum tensor, which includes the gravitational contribution. At the same time, doubts arise about the invariance of the rest mass. As will be shown in the following, the rest mass of a body seems to be of the universal kind of energy, i.e., to undergo a gravitational redshift. The two premises that support the geodesic path have therefore been affected. But certainly the geodesic path is strongly corroborated by the experimental facts related to the motion of the planets as well as to the falling of bodies. We would therefore be in serious trouble if our approach leads to perceptibly different results.

Whenever the interchange between matter and gravitational energy is negligible and the gravitational potential along the path almost constant, e.g., in the nearly circular motion of a planet, we are allowed to expect small differences between the path resulting from our approach and the geodesic path. But we are obligated to show that even in the case of a falling body its velocity, calculated by balancing matter and gravitational energy, satisfies the experimental data, i.e., approximates the geodesic path velocity.

To this purpose, let us now add to the spherically symmetric gravitational field of the previous section a spherical layer of dust of infinitesimal thickness dr. If the dust is at rest at infinity, the three integral values of the dust energy, equations (25)–(27), coincide with its rest mass at infinity, which will be denoted by dm. For the dust at rest at a finite Schwarzschild coordinate r the integrals split into three different values. They also will take different values for the universal and the local definition of energy-momentum, respectively. Choosing the universal definition of energy-momentum, it is conserved energy, the time component of a contravariant energy-momentum vector expressed by equation (25), that retains the value of the rest mass at infinity. Hence, one has in static coordinates at every point for the at-rest dust

$$dm^0 = dm = \int_{dV} (-g)^{1/2} T^{00}\, d^3x \tag{36}$$

where dV is the infinitesimal volume occupied by the dust. In contrast, the scalar mass of the at-rest dust, equation (26), which, had we used the local definition of energy-momentum, would have been an invariant of the motion, undergoes a gravitational redshift

$$\begin{gathered}\Phi_0\, dm^0 = \omega\, dm = \int_{dV} (-g)^{1/2} \Phi_0 T^{00}\, d^3x \\ \omega = (1 - 2MG/r)^{1/2}\end{gathered} \tag{37}$$

Consider now the energy of the dust layer falling from infinity. The contribution of the dust to the conserved energy of the system is given by

$$dm + dP^0 = \int_{dV} (-g)^{1/2} T^{00}\, d^3x \tag{38}$$

where dP^0 is the conserved kinetic energy of the dust. The contribution of the dust to the Schwarzschild mass parameter M, which determines the curvature as well as the gravitational energy in the outward region, is given by

$$dM = \omega^2\, dm + dP_0 = \int_{dV} (-g)^{1/2} T_0^0\, d^3x \tag{39}$$

and the scalar energy of the dust is

$$\omega\, dm + dP = \int_{dV} (-g)^{1/2} \Phi_0 T^{00}\, d^3x \tag{40}$$

with

$$dP = \omega\, dP^0 = \omega^{-1}\, dP_0 \tag{41}$$

According to the equivalence principle, the ratio of the scalar energy of the falling dust to its at-rest scalar mass is, by means of equations (37) and (40),

$$1 + dP/\omega\, dm = 1/(1-v^2)^{1/2} \tag{42}$$

where v is the dust velocity measured in the at-rest locally geodesic coordinates. Notice that since the left side of the above equation is a ratio of two energies it is independent of the way that the energy is measured.

When the dust layer moves from ∞ to r the Schwarzschild mass M included in the sphere of radius r increments by dM, equation (39). Hence, the conserved gravitational energy E_G^0 contained in the spatial volume from r to ∞, expressed by equation (32) with R replaced by r, undergoes a change given by

$$dE_G^0 = (\partial E_G^0(r, M)/\partial M)\, dM \tag{43}$$

According to the energy-momentum conservation law, valid in the assumed static nonrotating coordinates, this change in the gravitational energy should be opposite to the change in the conserved energy of the dust,

$$dP^0 = -dE_G^0 \tag{44}$$

From equations (41), (42), and (44) one obtains for the velocity of the falling dust measured in the at-rest locally geodesic coordinates

$$v = [1 - (\omega^2\, \partial E_G^0(r, M)/\partial M)^2]^{1/2} \tag{45}$$

The first thing to be noticed is that the velocity v is, as should be expected, independent of the mass dm of the dust. By developing equation (45) in powers of $r/2MG$ and disregarding second and higher powers, we obtain

$$v \approx 2^{1/2} MG/r = v_{\mathrm{E}} \tag{46}$$

which is the geodesic-path velocity v_{E} (as well as the Newtonian velocity if r is identified as the Newtonian distance) of a test body falling from infinity.

For a body falling from infinity to the surface of a regular star with $r/2MG = 2.32 \times 10^5$ one obtains

$$v = 0.0020761364\ldots$$

which agrees with the Einstenian value

$$v_{\mathrm{E}} = 0.0020761369\ldots$$

up to the seventh digit.

This result supports the proposed tensorial expression for the gravitational energy as well as the assumption that the nonrotating coordinates can be specialized to be adapted to the static nature of spacetime. In Figure 2

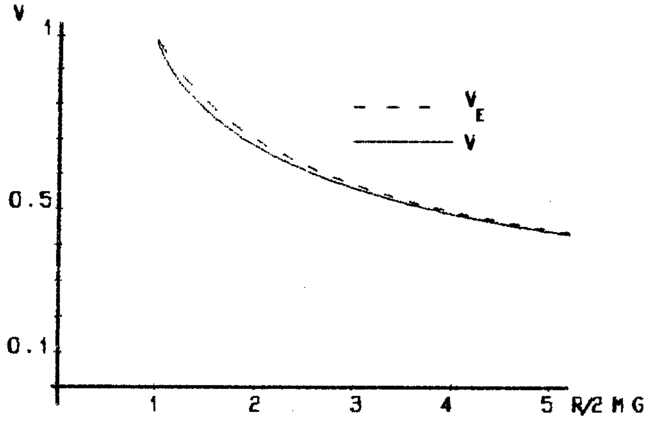

Fig. 2. The geodesic-path velocity V_E and the velocity V derived by balancing kinetic and gravitational energies, of a test body falling from infinity.

both velocities V and V_E of a body falling from infinity toward a black hole are plotted against $R/2MG$ for the last five Schwarzschild radii.

12. REMARKS AND CONCLUSIONS

A special class of coordinates, the nonrotating coordinates, is established by means of a variational principle imposed on the spacetime integrals of the components of the energy-momentum tensor. These preferred frames are therefore selected on account of their relation to the energy-momentum distribution in accordance with the Mach–Einstein assumption. Furthermore, they share the experimental properties of Newton's and Einstein's (special relativity) inertial frames:

1. They constitute the fixed-star frames wherein stars and galaxies are at rest or in constant-velocity motion.

2. They include locally-geodesic coordinates with respect to any given observer. Hence, they serve as adequate "inertial" frames for the local experiments that do not involve gravitation.

3. They also include non-locally-geodesic frames, making possible, at least in principle, the explanation of experimental facts that involve gravitation, e.g., the Newton water bucket and the Foucault pendulum experiments.

4. As it does in the inertial frames, the energy-momentum tensor satisfies in these coordinates a global conservation law.

5. In static spacetime, and with an appropriate tensorial definition of gravitational energy-momentum, they are adaptable to the static nature of spacetime.

6. The internal group of the geodesic nonrotating frames of a given observer is locally Lorentzian, in accordance with the central role that the Lorentz group plays in the physical phenomena that do not involve gravitation.

A tensorial expression for the gravitational energy-momentum is proposed. It is shown to be compatible with the existence of nonrotating coordinates adapted to the static characteristic of spacetime in the surroundings of a spherical source. The space integral of the gravitational energy in the space around a regular star results in high agreement with the Newtonian value. In the case of a star of solar dimensions the difference between the two values, the Newtonian and the relativistic, is of the order of one part in one million.

The study, by means of balancing gravitational against kinetic energy, of the falling of matter from infinity to the surface of a star gives results that accord with experience. The escape velocity from the sun calculated in this way matches the corresponding geodesic-path value to seven significant digits.

As a significant by-product of the investigation of the dynamics of matter in a gravitational field, we are led to distinguish between two definitions of energy-momentum. We denote the ensuing physical quantities the "local" and the "universal" energy-momentum. The first one is measured by comparison with a locally emitted photon, while the second is measured by comparison with a universal photon, for example, the background radiation. The first hides the gravitational redshift of energy (and frequency), whereas the second reveals it. Only by using the universal definition of energy-momentum can one obtain the correct results in calculating the falling velocity. This suggests that the energy-momentum tensor in the Einstein field equation has to be of universal nature.

REFERENCES

Carmeli, M., Leibowitz, E., and Nissani, N. (1990). *Gravitation: SL(2,C) Gauge Theory and Conservation Laws*, World Scientific, Singapore, p. 77.

Einstein, A., and Grommer, J. (1927). *Sitzungsberichte der Preussischen Academie der Wissenschaften Physik Mathematik Klasse*, **1**, 2.

Fock, V. (1964). *The Theory of Space Time and Gravitation*, Pergamon Press, Oxford, p. 369.

Infeld, L., and Schild, A. (1949). *Review of Modern Physics*, **21**, 408.

Landau, L. D., and Lifshitz, E. M. (1975). *The Classical Theory of Fields*, Pergamon Press, Oxford, p. 303.

Nissani, N., and Leibowitz, E. (1988). *Physics Letters A*, **126**, 447.

Nissani, N., and Leibowitz, E. (1989*a*). *International Journal of Theoretical Physics*, **28**, 235.

Nissani, N., and Leibowitz, E. (1989*b*). In *Proceedings of the Fifth Marcel Grossmann Meeting on General Relativity*, D. G. Blair and M. J. Buckingham, eds., World Scientific, Singapore, Vol. A, p. 415.

Nissani, N., and Leibowitz, E. (1990). In *Proceedings of the Third Canadian Conference on General Relativity and Relativistic Astrophysics*, World Scientific, Singapore.
Nissani, N., and Leibowitz, E. (1991). *International Journal of Theoretical Physics*, **30**, 837.